On the mechanics of a magnetohydrodynamical twin wake

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In this paper, we consider the elementary processes involved in the appearance of a simple twin wake in the two-dimensional steady aligned flow of an electrically conducting viscous incompressible fluid in a uniformly applied magnetic field. Fundamental relations between the twin wake and its elementary process are generally discussed, and the corresponding initial-value problem is formulated. Then, the solution of the initial-value problem corresponding to the above simple twin wake is obtained, and two limiting cases with the magnetic Prandtl number near one and zero (or infinite) are discussed in detail. In both cases, it is found that negative vorticity appears in between the main positive vorticity associated with the two Alfvén spots. This fact gives some clue to the understanding of the property of twin wakes. As an application, the crossed-flow case is also discussed, and a conjecture upon the vorticity distribution is made.

1. Introduction

In this paper, we shall restrict ourselves to the two-dimensional motion of an electrically conducting viscous incompressible fluid in a uniformly applied magnetic field. The study of steady small perturbations in an aligned uniform field is fundamental in magnetohydrodynamics and has attracted the attention of several authors (see, for example, Greenspan & Carrier 1959; Hasimoto 1959, 1960; Lary 1962; Yosinobu & Kakutani 1959; Yosinobu 1960). Among the characteristic features hitherto found, the most interesting for us is the appearance of a twin wake: one of these always extends to infinity downstream and the other to infinity up- or downstream as the Alfvén number is smaller or larger than one. The latter becomes thinner as the magnetic Prandtl number is decreased (or increased) from one, and disappears in the limit of zero (or infinite) Prandtl number, while the other retains a definite width even for the above limits.

By what mechanism do these wakes appear? And more especially, by what mechanism does a wake disappear in the limits? The first question seems, at first sight, self-evident from an intuitive point of view. Yosinobu assumed, in analogy with the classical diffusion process, that point vorticity shed instantaneously from the origin into the fluid at rest diffuses in the vicinity of the two Alfvén spots emanating from the origin with the Alfvén speed (figure 1), and answered the first question in the following manner: the wakes can be taken as the pattern of the diffusion process of the vorticity shed continuously from the origin into the uniform flow. This might seem to be a complete answer, but, as will be shown later, it is only complete when the magnetic Prandtl number is equal to one. Moreover, there remains the problem of why and how one can proceed on the basis of the analogy with the classical diffusion process. Thus, an apparently simple first question is in reality very difficult. The difficulty will become clearer if one considers the first question in connexion with the second.

In this paper, we consider the above two questions anew from the unified point of view that any phenomena in the steady flow can be explained by a corresponding elementary process which is universal with respect to the velocity of the uniform flow. In § 2, we consider the relations between the twin wake and its elementary process, and formulate the corresponding initial value problem. In § 3, the above questions with respect to a simple twin wake are considered and an elementary process including Yosinobu's assumption as a special case is obtained. Limiting cases with the magnetic Prandtl number near one and zero (or infinite) are discussed in detail, from which an intuitive answer can be given. As an application of the theory, the steady small perturbations in the crossed uniform flow are considered in § 4, and a conjecture upon the extensions of Hasimoto's result (Hasimoto 1960) to the case, when the magnetic Prandtl number is near one, is given.

2. Relations between the twin wake and its elementary process

In considering the elementary process, we assume that the disturbances in the fluid, which is initially at rest and in a uniform field $\mathbf{i}B_0$, are small. The equations governing the motion can then be linearized and in a M.K.S. system of units are

$$\widetilde{\mathrm{curl}}\,\tilde{\mathbf{e}} = -\partial\tilde{\mathbf{b}}/\partial\tilde{t}, \tag{2.1}$$

$$\tilde{\rho}_e = 0, \tag{2.2}$$

$$\widetilde{\mathrm{curl}}\,\tilde{\mathbf{b}} = \mu\tilde{\mathbf{j}}, \tag{2.3}$$

$$\widetilde{\mathrm{div}}\,\tilde{\mathbf{b}} = 0, \tag{2.4}$$

$$\tilde{\mathbf{j}} = \sigma(\tilde{\mathbf{e}} + \tilde{\mathbf{v}} \times \mathbf{i}_1 B_0), \tag{2.5}$$

$$\widetilde{\mathrm{div}}\,\tilde{\mathbf{v}} = 0, \tag{2.6}$$

$$\rho_0\,\partial\tilde{\mathbf{v}}/\partial\tilde{t} = -\widetilde{\mathrm{grad}}\,\tilde{p} + \rho_0\nu\tilde{\Delta}\tilde{\mathbf{v}} + \tilde{\mathbf{j}} \times \mathbf{i}_1 B_0, \tag{2.7}$$

where $\tilde{\mathbf{e}}$ is the electric field, $\tilde{\rho}_e$ the electric charge density, $\mathbf{i}_1 B_0 + \tilde{\mathbf{b}}$ the magnetic induction, $\tilde{\mathbf{j}}$ the electric current density, $\tilde{\mathbf{v}}$ the velocity, $p_0 + \tilde{p}$ the pressure, p_0 the undisturbed pressure, ρ_0 the density, μ the magnetic permeability, σ the electric conductivity, ν the kinematic viscosity, $(\tilde{x}, \tilde{y}, \tilde{z})$ the right-handed cartesian co-ordinates system with the $\tilde{x}$-axis in the direction of $\mathbf{i}_1$ and the tilde over each letter indicates that the letter is a variable in the fluid at rest, respectively. In the above equations, the assumption of two-dimensionality

$$\tilde{\mathbf{v}} = (\tilde{u}, \tilde{v}, 0), \quad \tilde{\mathbf{b}} = (\tilde{b}_{\tilde{x}}, \tilde{b}_{\tilde{y}}, 0), \quad \partial/\partial\tilde{z} = 0, \tag{2.8}$$

is used implicitly, so that $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{j}}$ must be understood to be of the following form:

$$\tilde{\mathbf{e}} = (0, 0, \tilde{e}), \quad \tilde{\mathbf{j}} = (0, 0, \tilde{j}). \tag{2.9}$$

Operating with $\widetilde{\text{curl}}$ on (2.5) and (2.7) and eliminating $\tilde{\mathbf{e}}$ by (2.1), we obtain

$$l_{\nu_m}(\tilde{x},\tilde{y},\tilde{t})\,\tilde{j} = -(B_0/\mu)\,\partial\tilde{\omega}/\partial\tilde{x}, \tag{2.10}$$

$$l_{\nu}(\tilde{x},\tilde{y},\tilde{t})\,\tilde{\omega} = -(B_0/\rho_0)\,\partial\tilde{j}/\partial\tilde{x}, \tag{2.11}$$

where $$l_\nu(\tilde{x},\tilde{y},\tilde{t}) = \nu\tilde{\Delta} - \partial/\partial\tilde{t}, \quad \tilde{\Delta} = \partial^2/\partial\tilde{x}^2 + \partial^2/\partial\tilde{y}^2,$$

and $\tilde{\omega} = \partial\tilde{v}/\partial\tilde{x} - \partial\tilde{u}/\partial\tilde{y}$ is the vorticity and $\nu_m = (\mu\sigma)^{-1}$ the magnetic diffusivity, respectively.

Eliminating $\tilde{j}$ (or $\tilde{\omega}$), we obtain

$$L(\tilde{x},\tilde{y},\tilde{t})\begin{pmatrix}\tilde{\omega}\\ \tilde{j}\end{pmatrix} = 0, \tag{2.12}$$

where $$L(\tilde{x},\tilde{y},\tilde{t}) = l_\nu(\tilde{x},\tilde{y},\tilde{t})\,l_{\nu_m}(\tilde{x},\tilde{y},\tilde{t}) - a_m^2\,\partial^2/\partial\tilde{x}^2,$$

and $a_m = (B_0^2/\mu\rho_0)^{\frac{1}{2}}$ is the Alfvén speed. Equations (2.10)–(2.12) reduce to those appropriate to the propagation of Alfvén waves in the special case when

$$\nu = \nu_m = 0;$$

the full equations are generally taken to represent the diffusion of the Alfvén waves.

Indeed the latter is the case at sufficiently large distances from the centre of the disturbance, as can be seen by putting equation (2.12) in the form:

$$\left(\frac{\partial}{\partial\tilde{t}} - a_m\frac{\partial}{\partial\tilde{x}} - \bar{\nu}\tilde{\Delta}\right)\left(\frac{\partial}{\partial\tilde{t}} + a_m\frac{\partial}{\partial\tilde{x}} - \bar{\nu}\tilde{\Delta}\right)\tilde{\omega} = -\tfrac{1}{4}(\nu-\nu_m)^2\,\tilde{\Delta}^2\tilde{\omega}, \tag{2.12a}$$

when $\bar{\nu} = \frac{1}{2}(\nu+\nu_m)$. At very large distances all gradients are small and the lowest order derivatives must predominate, so that the right-hand side of the above equation will be small. Neglecting the right-hand side, the resulting equation tells us that diffusion takes place about the two Alfvén waves with diffusivity equal to the algebraic mean of the magnetic diffusivity and viscosity. The same is true if the magnetic Prandtl number is unity, since the right-hand side disappears also in that case.

At distances closer to the origin of the disturbances it may be thought that the fourth order term in the right-hand side of equation (2.12 *a*) might contribute to the motion, and it is this effect which we seek to examine. Of necessity it must be stressed that the above equation is the result of linearizing the equations of motion, and though this might be true at large distances, it may not be true at smaller distances. However, there might well be a restricted region depending on the size and nature of the disturbance, where linearization is valid but where the right-hand side makes a definite contribution. It is in such a region that the results of this paper will be applicable. Since the peculiarity of such diffusion phenomena in incompressible fluid is expected to be clarified by the study of $\tilde{j}$ and $\tilde{\omega}$, we shall restrict ourselves to it. Further, if we restrict ourselves to the disturbances emanated from the finite part in the space, we require the following boundary conditions:

$$\tilde{j} \to 0 \quad \text{and} \quad \tilde{\omega} \to 0 \quad \text{as} \quad \tilde{x}^2 + \tilde{y}^2 \to \infty. \tag{2.13}$$

The linearized basic equations for the steady small perturbations in the uniform flow $\mathbf{v}_0 = (u_0, v_0, 0)$ can be obtained by applying the infinitesimal Galilei transformation

$$\tilde{\mathbf{e}} = \mathbf{e} + \mathbf{v}_0 \times \mathbf{b}, \tag{2.14}$$

$$\tilde{\rho}_e = \rho_e, \tag{2.15}$$

$$\tilde{\mathbf{b}} = \mathbf{b}, \tag{2.16}$$

$$\tilde{\mathbf{j}} = \mathbf{j}, \tag{2.17}$$

$$\tilde{\mathbf{v}} = \mathbf{v}, \tag{2.18}$$

$$\tilde{p} = p, \tag{2.19}$$

$$(\tilde{x}, \tilde{y}, \tilde{z}) = (x - u_0 t, y - v_0 t, z), \tag{2.20}$$

$$\tilde{t} = t, \tag{2.21}$$

on the above equations and neglecting the terms with $\partial/\partial t$. The equations for j and $\omega = \partial v/\partial x - \partial u/\partial y$ are as follows:

$$l_{\nu m}(x, y, \mathbf{x}/\mathbf{v}_0) j = -(B_0/\mu)\, \partial\omega/\partial x, \tag{2.22}$$

$$l_{\nu}(x, y, \mathbf{x}/\mathbf{v}_0)\, \omega = -(B_0/\rho_0)\, \partial j/\partial x, \tag{2.23}$$

$$L(x, y, \mathbf{x}/\mathbf{v}_0) \begin{pmatrix} \omega \\ j \end{pmatrix} = 0, \tag{2.24}$$

where $\mathbf{x}/\mathbf{v}_0$ in the operators indicates that $\partial/\partial\tilde{t}$ in the original operators is changed into $(\mathbf{v}_0 . \text{grad})$.

Since the above equations are linear, j and ω can be obtained by the superposition of $\tilde{j}$ and $\tilde{\omega}$ as follows:

$$j = \int_0^\infty \tilde{j}(x - u_0 t, y - v_0 t, t)\, dt, \tag{2.25}$$

$$\omega = \int_0^\infty \tilde{\omega}(x - u_0 t, y - v_0 t, t)\, dt. \tag{2.26}$$

The boundary conditions (2.13) are then transformed as follows:

$$j \to 0 \quad \text{and} \quad \omega \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{2.27}$$

These equations are the mathematical expression of our intuitive expectations that the steady perturbations can be constructed by the unsteady ones shed continuously into the uniform flow and that each of the latter reduces to that in the fluid at rest if observed on the basis of a co-ordinate system moving with the uniform flow. Substituting these into the left-hand side of (2.22) and (2.23) and taking into account (2.10) and (2.11) and assuming that $\tilde{j} \to 0$ and $\tilde{\omega} \to 0$ as $\tilde{t} \to \infty$, we obtain

$$l_{\nu m}(x, y, \mathbf{x}/\mathbf{v}_0) j + (B_0/\mu)\, \partial\omega/\partial x = -\tilde{j}(x, y, 0), \tag{2.28}$$

$$l_{\nu}(x, y, \mathbf{x}/\mathbf{v}_0)\, \omega + (B_0/\rho_0)\, \partial j/\partial x = -\tilde{\omega}(x, y, 0). \tag{2.29}$$

The steady perturbations constructed as above have the source corresponding to the initial perturbations, as will be expected from their construction. On the other hand, since the equations of the form (2.28) and (2.29) are those of the singular solution representing the twin wake, the above can be taken as showing the one-to-one correspondence between the twin wake and the initial perturbation. The initial perturbation may depend on $\mathbf{v}_0$. However, since the range of its values can be covered by the family of it independent of $\mathbf{v}_0$, we shall restrict ourselves to the family, hereafter. This restriction represents our point of view that any phenomena in the steady flow can be explained by the corresponding elementary process universal with respect to $\mathbf{v}_0$.

On the basis of the results obtained above, we shall discuss the unsteady formation of a simple twin wake satisfying the singular equations (2.28) and (2.29) and the boundary conditions (2.27) by means of the solution of (2.10) and (2.11) satisfying the boundary conditions (2.13) and the initial conditions

$$\tilde{j} = \tilde{j}(\tilde{x}, \tilde{y}, 0) \quad \text{and} \quad \tilde{\omega} = \tilde{\omega}(\tilde{x}, \tilde{y}, 0) \quad \text{for} \quad \tilde{t} = 0, \tag{2.30}$$

where the right-hand sides are independent of $\mathbf{v}_0$. Since the initial values of $\tilde{j}_{\tilde{t}}$ and $\tilde{\omega}_{\tilde{t}}$ can be obtained from those of $\tilde{j}$ and $\tilde{\omega}$, or alternatively those of $\tilde{j}$ and $\tilde{j}_{\tilde{t}}$ by those of $\tilde{\omega}$ and $\tilde{\omega}_{\tilde{t}}$ (see (2.10) and (2.11)), the above initial-value problem is equivalent to the problem of solving (2.12) with respect to $\tilde{\omega}$ under the boundary condition (2.13) and the initial conditions

$$\tilde{\omega} = \tilde{\omega}(\tilde{x}, \tilde{y}, 0) \quad \text{and} \quad \tilde{\omega}_{\tilde{t}} = \tilde{\omega}_{\tilde{t}}(\tilde{x}, \tilde{y}, 0) \quad \text{for} \quad \tilde{t} = 0. \tag{2.31}$$

Therefore, we shall further restrict ourselves to the consideration of $\tilde{\omega}$ and discuss only the vorticity wake by the solution of the above form of initial-value problem.

3. Elementary process of a simple twin wake

Let us consider a small perturbation in the aligned uniform flow $\mathbf{v}_0 = (u_0, 0, 0)$ represented as follows:

$$\omega_0 = \sum_{i=1}^{2} \omega_{0i}(x, y), \tag{3.1}$$

where

$$\omega_{0i} = (2\pi)^{-1} \exp\left(\tfrac{1}{2}k_i x\right) \mathrm{K}_0\left(\tfrac{1}{2}|k_i|\, r\right), \quad r = (x^2+y^2)^{\frac{1}{2}}, \tag{3.2}$$

$$k_i = [u_0(\nu+\nu_m) - (-1)^i \{u_0^2(\nu+\nu_m)^2 - 4\nu\nu_m(u_0^2 - a_m^2)\}^{\frac{1}{2}}]/2\nu\nu_m, \tag{3.3}$$

and K_0 is the modified Bessel function. As has been shown by previous authors, ω_{0i} is nearly zero almost all over the flow field at a great distance from the origin except in the wake near the x-axis, extending to infinity down- or upstream as k_i is positive or negative, and having a width roughly proportional to $\{|x|/|k_i|\}^{\frac{1}{2}}$. Thus, ω_0 represents a simple twin wake one part of which extends always to infinity downstream while the other to infinity down- or upstream as the Alfvén number $A = u_0/a_m$ is larger or smaller than 1 (figure 2). ω_0 is symmetric with respect to ν and ν_m, and the wake represented by ω_{01} becomes thinner as the magnetic Prandtl number $P_m = \nu_m/\nu$ is decreased (or increased) from 1 and disappears in the limit of zero (or infinite) P_m while the other retains a definite width even in these limits. Since the widths of the wakes in the case with $P_m = 1$ are proportional to $\{\nu\,|x|/(u_0+a_m)\}^{\frac{1}{2}}$ and $\{\nu\,|x|/|u_0-a_m|\}^{\frac{1}{2}}$, we can expect an elementary process similar to that assumed by Yosinobu (figure 1). Since the parts in the general width $\{|x|/|k_i|\}^{\frac{1}{2}}$ corresponding to u_0 and a_m are proportional to u_0 and $\{u_0^2(\nu+\nu_m)^2 - 4\nu\nu_m(u_0^2 - a_m)\}^{\frac{1}{2}}$, we must make an extraordinary assumption about the speed of emanation of the centre of the diffusion if we want still to accept Yosinobu's assumption for the general case with $P_m \neq 1$. However, such a fact is nothing but the indication of the impropriety of the above assumption.

Now, it can be shown by the method of Fourier integrals that the above simple twin wake satisfies the following singular equation:

$$L(x, y, x/u_0)\,\omega_0 = -\{2\nu\nu_m \Delta - u_0(\nu+\nu_m)\,\partial/\partial x\}\,\delta(x, y), \tag{3.4}$$

where δ is Dirac's delta function. Comparing this with

$$L(x, y, x/u_0)\,\omega_0 = \{-(\nu+\nu_m)\,\tilde{\Delta}\tilde{\omega}_0 + \partial\tilde{\omega}_0/\partial\tilde{t} + u_0\,\partial\tilde{\omega}_0/\partial\tilde{x})|_{\tilde{t}=0}, \tag{3.5}$$

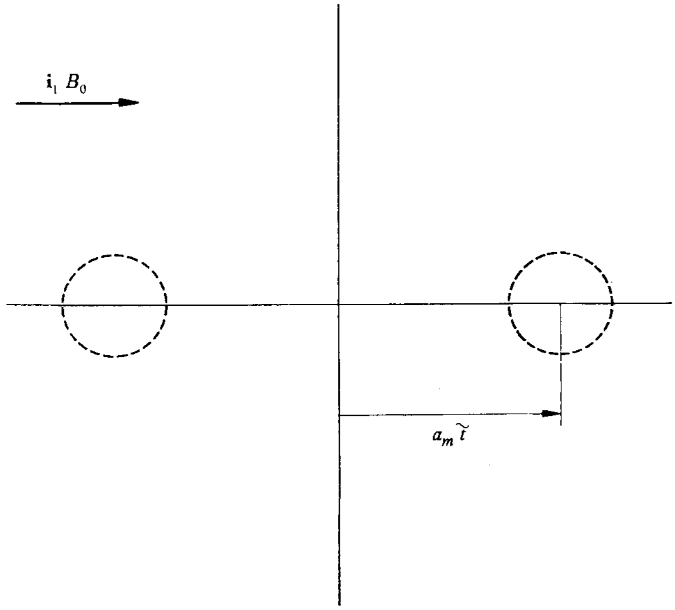

FIGURE 1. Yosinobu's assumption for the elementary process of MHD twin-wake formation.

obtained by the elimination of j from (2.28) and (2.29), we obtain the following initial conditions for the elementary process of the above twin wake:

$$\tilde{\omega}_0(\tilde{x}, \tilde{y}, 0) = (\nu+\nu_m)\,\delta(\tilde{x}, \tilde{y}), \tag{3.6}$$

$$\partial\tilde{\omega}_0/\partial\tilde{t}(\tilde{x}, \tilde{y}, 0) = (\nu^2+\nu_m^2)\,\tilde{\Delta}\delta(\tilde{x}, \tilde{y}), \tag{3.7}$$

where use is made of the fact that $\tilde{\omega}_0$ is independent of u_0 and that $\tilde{x} = x$ and $\tilde{y} = y$ at $\tilde{t} = 0$. As is expected from the property of ω_0, the basic equations of the above elementary process ((2.12), (2.13), (3.6) and (3.7)) are symmetric with respect to ν and ν_m. Since the solution for the case with $P_m > 1$ can be obtained from that for $P_m < 1$ by the exchange of ν and ν_m, we can restrict ourselves to the latter case. Similar restriction to a consideration of the first quadrant can be effected because of the symmetry with respect to $\tilde{x}$ and $\tilde{y}$.

The solution of the previous initial-value problem with the above initial conditions can be obtained by the method of Fourier integrals. However, since the calculations involved are straightforward and lengthy, only the results are given:

$$\tilde{\omega}_0 = \tilde{\omega}_1 + \tilde{\omega}_{2,1} + \tilde{\omega}_{2,2}, \tag{3.8}$$

$$\tilde{\omega}_1 = \frac{(\nu+\nu_m)\,a_m}{4\pi}\int_0^\infty \frac{\tilde{\Omega}_1 z\,dz}{\{\zeta_1^{\frac{1}{2}}(z^2+1)\}}, \tag{3.9}$$

$$\tilde{\Omega}_1 = \sum_{i=1}^{2} \exp(-R_{11i})\,[\{a_m^{-1}\tilde{t}^{-1}(z^2+1)^{-\frac{1}{2}} + X_{11i}\}\,\mathrm{J}_0(R_{12i}) + X_{12i}\mathrm{J}_1(R_{12i})], \tag{3.10}$$

where

$$\left.\begin{aligned} \zeta_1 &= (\nu+\nu_m)^2 + (\nu-\nu_m)^2 z^2,\\ x_{1i} &= a_m\tilde{t}(z^2+1)^{\frac{1}{2}} - (-1)^i\tilde{x},\\ X_{1ij} &= \eta_{1i}x_{1j}\tilde{t}^{-1},\\ \eta_{1i} &= (\nu+\nu_m)\,\zeta_1^{-1}\{|\nu-\nu_m|\,z(\nu+\nu_m)^{-1}\}^{i-1},\\ R_{1ij} &= \eta_{1i}(x_{1j}^2+\tilde{y}^2)\,2^{-1}\tilde{t}^{-1}, \end{aligned}\right\} \tag{3.11}$$

$$\tilde{\omega}_{21} = (\nu-\nu_m)^2\,(8\pi\tilde{t})^{-1}\int_0^1 \Omega_{21}\,\zeta_2^{-\frac{1}{2}}\,d\xi, \tag{3.12}$$

$$\begin{aligned}\tilde{\Omega}_{21} = \sum_{i=1}^{2} \exp(-R_{21i})\,[&\eta_{22}R_{22i}\{\mathrm{J}_2(R_{22i}) - \mathrm{J}_0(R_{22i})\}\\ &+ 2\eta_{21}(R_{21i}-1)\,\mathrm{J}_0(R_{22i}) + 2\eta_{22}(2R_{21i}-1)\,\mathrm{J}_1(R_{22i})],\end{aligned} \tag{3.13}$$

$$\tilde{\omega}_{22} = -(\nu+\nu_m)(\nu-\nu_m)^2\,(4\pi\tilde{t})^{-1}\int_0^\infty \zeta_2^{-\frac{3}{2}}\,\eta_{22}^{-1}\left\{\frac{\partial\tilde{\Omega}_{22}}{\partial\eta_{22}}\right\}d\xi, \tag{3.14}$$

$$\begin{aligned}\Omega_{22} = \sum_{i=1}^{2} \exp(-R_{21i})\,[&\eta_{22}^2\,2^{-1}\{\mathrm{J}_2(R_{22i}) - \mathrm{J}_0(R_{22i})\}\\ &+ 2\eta_{21}\eta_{22}\mathrm{J}_1(R_{22i}) + \eta_{21}^2\mathrm{J}_0(R_{22i})],\end{aligned} \tag{3.15}$$

where

$$\left.\begin{aligned} \zeta_2 &= (\nu+\nu_m)^2 + (\xi^2-1)(\nu-\nu_m)^2,\\ r_{2i}^2 &= \{a_m\tilde{t}\xi - (-1)^i\tilde{x}\}^2 + \tilde{y}^2,\\ \eta_{2i} &= (\nu+\nu_m)\,\zeta_2^{-1}\{(\xi^2-1)^{\frac{1}{2}}\,|\nu-\nu_m|\,(\nu+\nu_m)^{-1}\}^{i-1},\\ R_{2ij} &= \eta_{2i}r_{2j}^2(2\tilde{t})^{-1}, \end{aligned}\right\} \tag{3.16}$$

and the J_ν are ordinary Bessel functions. In the course of the calculations, we use the formula of Sonine and Gegenbauer:

$$\begin{aligned}&\int_0^\infty \mathrm{J}_\mu(bt)\,\mathrm{H}_\nu^{(2)}(a(t^2+x^2)^{\frac{1}{2}})\,(t^2+x^2)^{-\frac{1}{2}\nu}\,t^{\mu+1}\,dt\\ &= \begin{cases}(b^\mu/a^\nu)\,\{(a^2-b^2)^{\frac{1}{2}}/x\}^{\nu-\mu-1}\,\mathrm{H}_{\nu-\mu-1}^{(2)}(x(a^2-b^2)^{\frac{1}{2}}) & \text{for } a > b,\\ (2ib^\mu/\pi a^\nu)\,\{(b^2-a^2)^{\frac{1}{2}}/x\}^{\nu-\mu-1}\,\mathrm{K}_{\nu-\mu-1}(x(b^2-a^2)^{\frac{1}{2}}) & \text{for } a < b;\end{cases}\end{aligned} \tag{3.17}$$

the formula of Laplace transformation:

$$\begin{aligned}&(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}S^{-\nu}\mathrm{K}_\nu(bS)\,e^{pt}\,dp\\ &\qquad = \begin{cases}0 & \text{for } 0 < t < b,\\ a^{\frac{1}{2}-\nu}b^{-\nu}y^{\nu-\frac{1}{2}}\,\mathrm{I}_{\nu-\frac{1}{2}}(ay) & \text{for } t > b,\end{cases}\end{aligned} \tag{3.18}$$

where $S = (p^2-a^2)^{\frac{1}{2}}, \quad y = (t^2-b^2)^{\frac{1}{2}};$

the formulae for the integral representation of the Bessel functions:

$$\mathrm{J}_n(x) = 2(x/2)^{-n}\,\pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}-n)\int_1^{\infty}(t^2-1)^{-(n+\frac{1}{2})}\sin xt\,dt, \tag{3.19}$$

$$\mathrm{K}_n(ax) = (2x/a)^n\,\Gamma(n+\tfrac{1}{2})\,\pi^{-\frac{1}{2}}\int_0^{\infty}(t^2+x^2)^{-(n+\frac{1}{2})}\cos at\,dt, \tag{3.20}$$

$$\mathrm{J}_0(x) = (2/\pi)\int_0^1(1-t^2)^{-\frac{1}{2}}\cos xt\,dt; \tag{3.21}$$

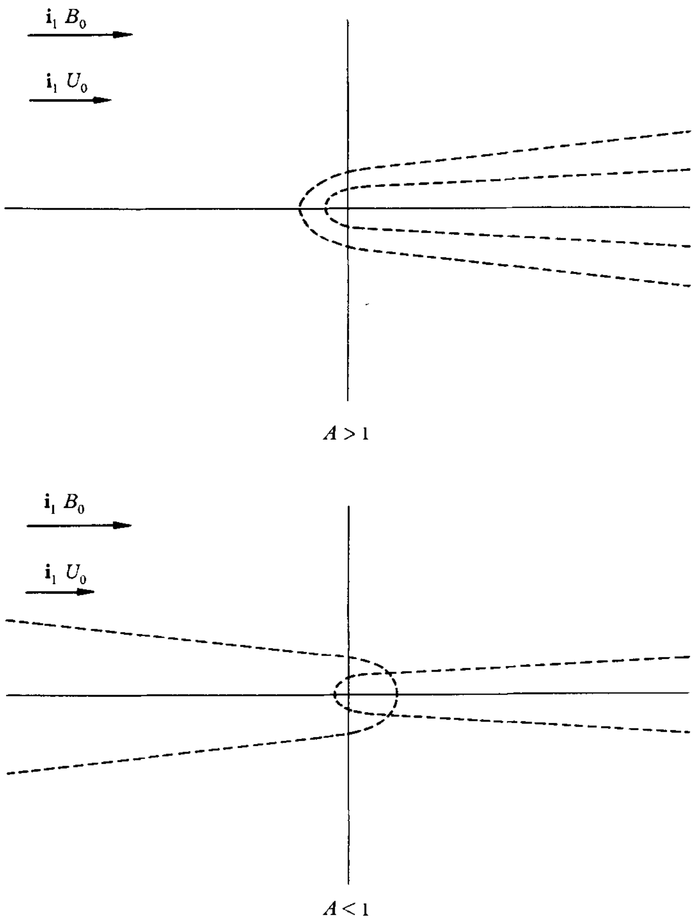

FIGURE 2. MHD twin wake.

the formula of the Fourier integral:

$$\int_0^{\infty} p^{2n}\exp(-a^2p^2)\cos px\,dp = (-1)^n\,\pi^{\frac{1}{2}}\,2^{-n-1}\,a^{-2n-1}\exp(-x^2/4a^2)\,\mathrm{He}_{2n}(2^{-\frac{1}{2}}x/a) \tag{3.22}$$

(see, for example, Magnus & Oberhettinger 1948 and Erdélyi 1954), the usual technique of the contour integral and assume that the order of the integrations can freely be exchanged.

Since it is prohibitively difficult to perform the above integrations for the general case, we restrict ourselves to the two typical limiting cases: (1) $P_m \doteqdot 1$ and (2) $P_m = 0$ (or $P_m = \infty$).

(1) *Case with* $P_m \doteqdot 1$. By expanding $\tilde{\omega}_0$ with respect to $|\nu-\nu_m|$ and retaining the terms up to the order $O(|\nu-\nu_m|^2)$, we obtain the following result:

$$\begin{aligned}\tilde{\omega}_0 = (4\pi\tilde{t})^{-1} \sum_{i=1}^{2} \exp\{-\tilde{r}_i^2/2\tilde{t}(\nu+\nu_m)\} \\ -(\nu-\nu_m)^2(16a_m)^{-1}\{2\pi\tilde{t}^3(\nu+\nu_m)^3\}^{-\frac{1}{2}}\exp\{-\tilde{y}^2/2\tilde{t}(\nu+\nu_m)\} \\ \times\Bigg[\{1+2\tilde{y}^2/\tilde{t}(\nu+\nu_m)-\tilde{y}^4/\tilde{t}^2(\nu+\nu_m)^2\} \\ \times\left\{1-2^{-1}\sum_{i=1}^{2}\operatorname{erf}c\,(\tilde{x}_i/\{2\tilde{t}(\nu+\nu_m)\}^{\frac{1}{2}})\right\} \\ -\{2\pi\tilde{t}(\nu+\nu_m)\}^{-\frac{1}{2}}\sum_{i=1}^{2}\tilde{x}_i\{1-(\tilde{x}_i^2+2\tilde{y}^2)/\tilde{t}(\nu+\nu_m)\}\exp\{-\tilde{x}_i^2/2\tilde{t}(\nu+\nu_m)\}\Bigg], \end{aligned} \quad (3.23)$$

where
$$\tilde{x}_i = a_m\tilde{t}-(-1)^i\tilde{x}, \quad \tilde{r}_i^2 = \tilde{x}_i^2+\tilde{y}^2. \quad (3.24)$$

When $P_m = 1$, the vorticity diffuses in the vicinity of two Alfvén spots emanating from the origin with the Alfvén speed showing the correctness of Yosinobu's assumption. However, if P_m differs slightly from 1, these main vorticities behave as if ν becomes the algebraic mean of ν and ν_m, and further there appears the region of negative vorticity diffusing roughly uniformly in between these main vorticities (figure 3).

Substituting (3.23) into (2.26), we obtain

$$\begin{aligned}\omega_0 \doteqdot (2\pi)^{-1}\sum_{i=1}^{2}\exp[x\{u_0-(-1)^i a_m\}/(\nu+\nu_m)]\,\mathrm{K}_0\{r\,|u_0-(-1)^i a_m|/(\nu+\nu_m)\} \\ -(\nu-\nu_m)^2\{4\pi a_m(\nu+\nu_m)^3\}^{-1} \\ \times\sum_{i=1}^{2}\{u_0-(-1)^i a_m\}^2\exp[x\{u_0-(-1)^i a_m\}/(\nu+\nu_m)]\,(-1)^i \\ \times[x\mathrm{K}_0\{r\,|u_0-(-1)^i a_m|/(\nu+\nu_m)\} \\ -(r\{u_0-(-1)^i a_m\}/|u_0-(-1)^i a_m|)\,\mathrm{K}_1\{r\,|u_0-(-1)^i a_m|/(\nu+\nu_m)\}], \end{aligned} \quad (3.25)$$

where $\mathbf{v}_0 = (u_0, 0, 0)$ has been used. The above agrees completely with the expansion of (3.1) with respect to $|\nu-\nu_m|$ up to the order $O[(\nu-\nu_m)^2]$ showing that our procedure is perfectly correct. Applying the asymptotic expansion of K_n to (3.25), we can show that the strength of both wakes is weakened and the degree of the weakening is larger for ω_{01}. This fact can be understood as the result of the cancelling action of the negative vorticity.

(2) *Case with* $P_m = 0$ (*or* $P_m = \infty$). In this case, we cannot carry out the simple expansion and thus cannot obtain a unified analytical expression as above. We therefore restrict ourselves first to the asymptotic behaviour of the vorticity distribution on the positive part of the $\tilde{x}$- and the $\tilde{y}$-axis for large $\tilde{t}$, and then conjecture upon its overall behaviour. By applying the usual method of calculating the asymptotic expression for the integral, we obtain the following results.

On the $\tilde{x}$-axis:

$$\tilde{\omega}_0 \doteqdot \nu^{\frac{1}{2}}(4\pi a_m \tilde{t}^{\frac{3}{2}})^{-1} \times \left[\pi^{-\frac{1}{2}} + 2^{\frac{1}{2}} \int_0^2 \{\mathrm{I}_1(s) - \mathrm{I}_0(s)\}\, e^{-s} s^{\frac{1}{2}}\, ds - (2\pi^{\frac{1}{2}})^{-1} \int_{\tilde{x}^2(a_m \tilde{t})^{-2}}^{\infty} e^{-\zeta} \zeta^{-1}\, d\zeta\right] \quad \text{for} \quad \tilde{x} \sim O(\tilde{t}^{\frac{1}{2}}), \tag{3.26}$$

$$\tilde{\omega}_0 \doteqdot \nu^{\frac{1}{2}}(4\pi a_m \tilde{t}^{\frac{3}{2}})^{-1} \{2^{\frac{1}{2}} a_m^2 \tilde{t}^2 (1-k^2)\, \tilde{x}^{-2} k^{-2-\frac{1}{2}}\} \times \left[(2\pi)^{-\frac{1}{2}} \int_{2k}^{\infty} \{1 - k - (3+k)/8S \ldots\} \exp\{-(1-k)\, S/k\}\, dS + \int_0^{2k} \{\mathrm{I}_1(S) - k\mathrm{I}_0(S)\}\, S^{\frac{1}{2}} \exp(-S/k)\, dS\right] \quad \text{for} \quad \begin{matrix} \tilde{x} \sim O(\tilde{t}), \\ \tilde{x} < a_m \tilde{t}, \end{matrix} \tag{3.27}$$

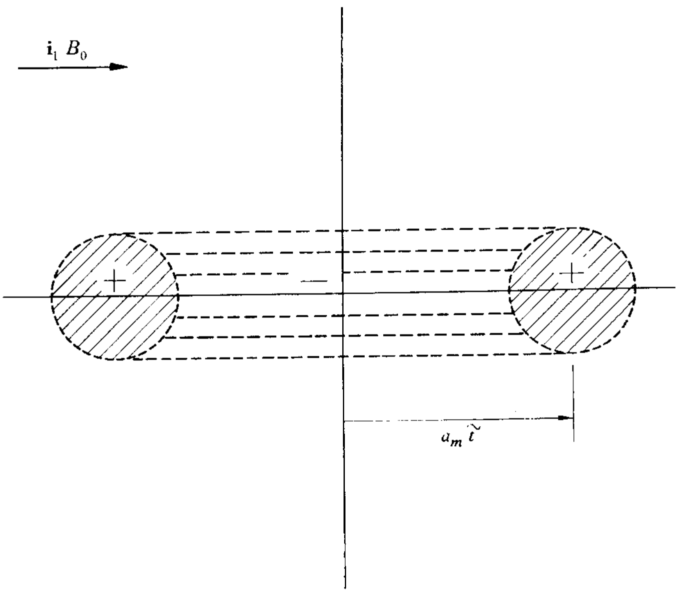

FIGURE 3. The elementary process in the case with $P_m \doteqdot 1$.

where $$k = \{1 - (\tilde{x}/a_m \tilde{t})^2\}^{\frac{1}{2}},$$

$$\tilde{\omega}_0 \doteqdot (4\pi\tilde{t})^{-1} a_m^2 \tilde{t}^2 \tilde{x}^{-2} \exp(-\kappa^2) - \nu^{\frac{1}{2}}(4\pi a_m \tilde{t}^{\frac{3}{2}})^{-1} [\{(2\pi)^{\frac{1}{2}}/16\}\{1 + \mathrm{erf}(\kappa)\} + 2^{\frac{1}{2}} 8^{-1} \kappa (15 + 2\kappa^2) \exp(-\kappa^2)] \quad \text{for} \quad \tilde{x} \sim a_m \tilde{t}, \tag{3.28}$$

where $$\kappa = (a_m \tilde{t} - \tilde{x})/(2\nu\tilde{t})^{\frac{1}{2}},$$

$$\omega_0 \doteqdot \nu^{\frac{1}{2}}(4\pi a_m \tilde{t}^{\frac{3}{2}})^{-1} (2\pi)^{\frac{1}{2}} (\tilde{x}^2/a_m^2 \tilde{t}^2) \{2\mathrm{F}(\tfrac{1}{4}, \tfrac{3}{4};\, 1;\, -K^2) - 2\mathrm{F}(\tfrac{3}{4}, \tfrac{5}{4};\, 1;\, -K^2) + \tfrac{3}{8}(1 - 3K^2)\, \mathrm{F}(\tfrac{5}{4}, \tfrac{7}{4};\, 2;\, -K^2)\} \quad \text{for} \quad \begin{matrix} \tilde{x} \sim O(\tilde{t}), \\ \tilde{x} > a_m \tilde{t}, \end{matrix} \tag{3.29}$$

where $$K = \{\tilde{x}^2 (a_m \tilde{t})^{-2} - 1\}^{\frac{1}{2}},$$

and F is the hypergeometric function.

On the $\tilde{y}$-axis:

$$\tilde{\omega}_0 \doteqdot -\nu^{\frac{1}{2}}(4\pi a_m \tilde{t}^{\frac{3}{2}})^{-1}\, 2^{-1} \exp\left(-\tilde{y}^2/8\nu\tilde{t}\right) \times \left[(4\nu\tilde{t}/\tilde{y}^2)^{\frac{1}{2}}\, \mathrm{W}_{0,0}(\tilde{y}^2/4\nu\tilde{t}) - 2\mathrm{W}_{\frac{1}{2},\frac{1}{2}}(\tilde{y}^2/4\nu\tilde{t})\right], \tag{3.30}$$

where W is the Whittaker function. The vorticity distribution on the $\tilde{x}$-axis is shown in figure 4, and the conjecture upon the asymptotic behaviour of the overall vorticity distribution is given in figure 5.

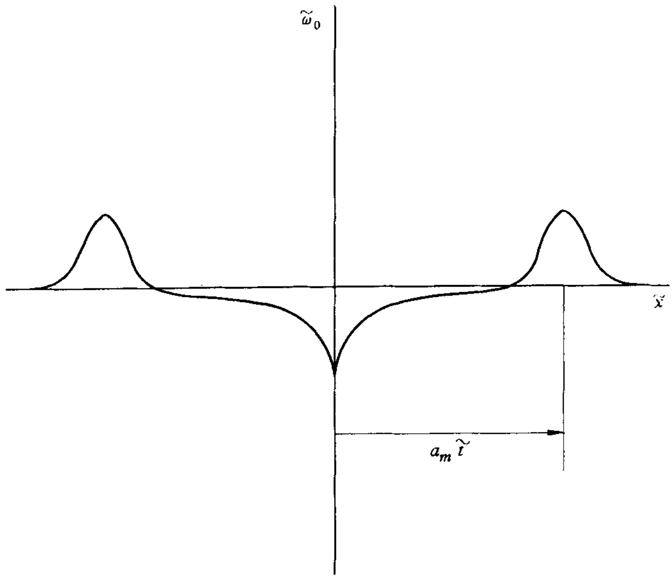

FIGURE 4. Vorticity distribution on the $\tilde{x}$-axis in the case with $P_m = 0$ (or ∞).

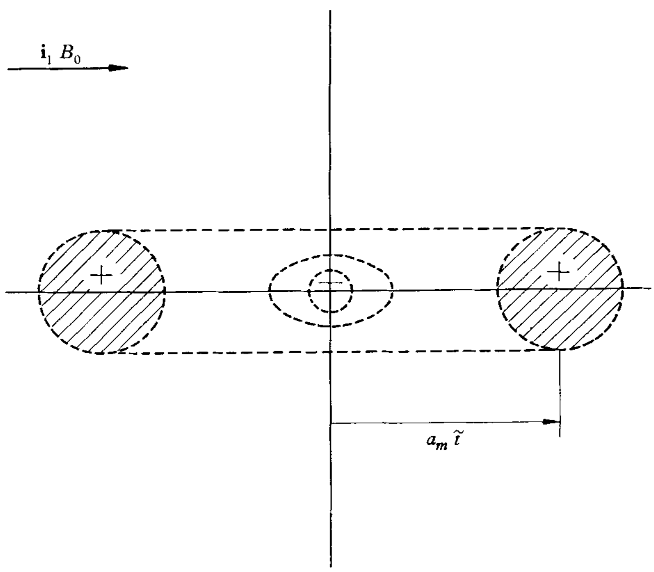

FIGURE 5. The elementary process in the case with $P_m = 0$ (or ∞).

As in the previous case, the main positive vorticity diffuses in the vicinity of two Alfvén spots emanating from the origin with the Alfvén speed, and the negative vorticity diffuses in between these. An important difference between these limiting cases consists in the fact that the negative vorticity now concentrates in the vicinity of the origin. Thus the disappearance of a wake in this limit can be understood as the result of the complete cancellation of the rear wake ω_{01} by the interference between a part of the main vorticity and the negative vorticity.

The above expected cancellation of wakes is obtained from a consideration of a special simple case. However, the fact that the wakes cancel because of the interference between the vorticity regions with opposite sign will be the key to understanding the property of magnetohydrodynamical twin wakes.

4. A conjecture upon the vorticity distribution in a uniform flow crossed by the magnetic field

The results in the previous section were obtained with special regard to the unsteady formation of a simple twin wake in the aligned uniform flow. However, since they are universal with respect to the velocity of the uniform flow,

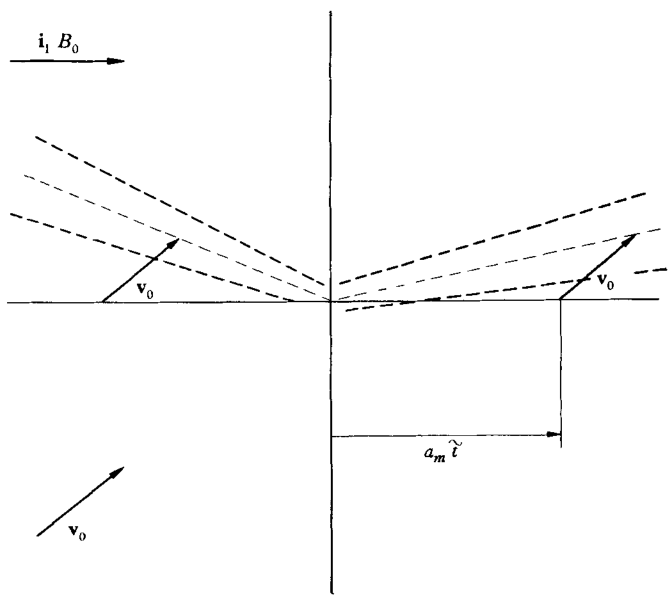

FIGURE 6. An oblique twin wake in the case with $P_m = 1$.

we can apply them to the case where the uniform flow is crossed by the applied magnetic field. In fact, they correspond to the steady vorticity field satisfying the following singular equation (see (2.28) and (2.29)):

$$L(x, y, \mathbf{x}/\mathbf{v}_0)\,\omega = -2\nu\nu_m \Delta\delta + (\nu+\nu_m)(\mathbf{v}_0 \cdot \mathrm{grad})\,\delta, \tag{4.1}$$

and the boundary condition:

$$\omega \to 0 \quad \text{as} \quad x^2+y^2 \to \infty. \tag{4.2}$$

These equations are solved by Hasimoto (1960) for the case $P_m = 1$, and the solution is:

$$\omega = \sum_{i=1}^{2} \omega_i(x,y), \tag{4.3}$$

$$\omega_i = (2\pi)^{-1} \exp\left([\{u_0 - (-1)^i a_m\} x + v_0 y]/2\nu\right) \mathrm{K}_0(r[\{u_0 - (-1)^i a_m\}^2 + v_0^2]^{\frac{1}{2}}/2\nu). \tag{4.4}$$

This shows that an oblique twin wake appears in the crossed uniform flow (figure 6). The extension of this result to the case of $P_m \doteqdot 1$ is an interesting problem; however, no positive results have yet been obtained. On the basis of our previous results (although we cannot obtain a simple analytical expression) we can easily arrive at a conjecture upon the vorticity distribution as shown in figure 7.

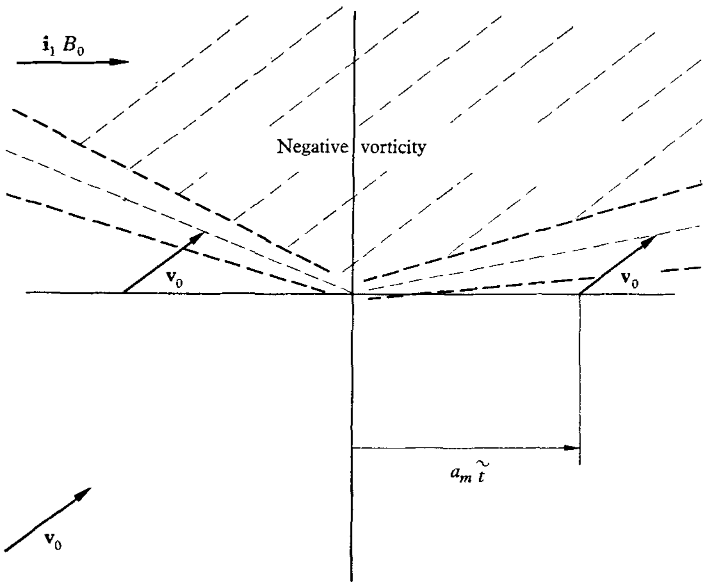

FIGURE 7. A conjecture upon the vorticity distribution in the crossed uniform flow case with $P_m \dot{\sim} 1$.

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